Hence by addition,

$$W(x_1)+W(1/x_1)=0. \tag{10}$$

For the moment, set $V(x_1)=W(x_1)\div(x_1-1/x_1)$. Then $V(1/x_1)=V(x_1)$. Hence $V(x_1)$ equals a rational function of $v=x_1+1/x_1$. But, by (1),

$$v^2-av+b-2=0. \tag{11}$$

The latter is irreducible in R since (1) is irreducible. Hence any rational function of v can be reduced to a linear function. Thus $V(x_1)$ may be given the form $lv+m$, so that

$$W(x_1)=(x_1-1/x_1)(lv+m). \tag{12}$$

The square of the first factor equals v^2-4. Applying (11), we may express W^2 and F, given by (8), as linear functions of v. The latter must be identically equal since (11) is irreducible. Hence

$$l^2(a^3-2ab)+2lm(a^2-b-2)+m^2a=-a, \tag{13}$$

$$l^2(2a^2-a^2b+b^2-4)+2lm(2a-ab)-m^2(b+2)=a^2-b-2. \tag{14}$$

Replacing x_1 by x_3 in (12), we get

$$W(x_3)=(x_3-1/x_3)[l(x_3+1/x_3)+m].$$

Applying (7), (9_2), and (12) to the latter, we get

$$W(x_3)=(x_1-1/x_1)(lv+m)(la+m-lv). \tag{15}$$

First, let the group be C_4. Apply (1324) to (9_2). Thus

$$x_2-1/x_2=W(x_3), \text{ or } 1/x_1-x_1=W(x_3). \tag{16}$$

Inserting the latter value of $W(x_3)$ in (15), we get

$$(lv+m)(la+m-lv)=-1.$$

Eliminating v^2 by means of (11), we get

$$l^2(b-2)+lma+m^2=-1. \tag{17}$$

The determinant of the coefficients of l^2, lm, m^2 in (13), (14), (17) equals $-4AB$ and hence is not zero (§ 1). The unique set of solutions is

$$l^2=\frac{\alpha^2}{4AB},\quad lm=\frac{-\alpha\alpha\beta}{4AB},\quad m^2=\frac{\alpha^2\beta^2}{4AB},$$
$$\alpha\equiv a^2-2b-4,\quad \beta\equiv a^2-3b+2.$$

Since l and m are rational, $\sqrt{(AB)}$ must be rational. This necessary condition for the group C_4 is also sufficient. Indeed, we get

$$l=\tfrac{1}{2}\alpha/\sqrt{(AB)},\quad m=-\tfrac{1}{2}a\beta/\sqrt{(AB)},$$

and all the roots are rational functions of x_1:

$$x_3=\tfrac{1}{2}[a-x_1-\frac{1}{x_1}+W(x_1)],\quad x_4=\tfrac{1}{2}[a-x_1-\frac{1}{x_1}-W(x_1)], \tag{18}$$

$$W(x_1)=\frac{1}{2\sqrt{(AB)}}(x_1-\frac{1}{x_1})[(a^2-2b-4)(x_1+\frac{1}{x_1})-a^3+3ab-2a]. \tag{19}$$

The Galois group, for a domain R, of an irreducible equation (1) *is the cyclic group C_4 if, and only if, $\sqrt{(AB)}$ is rational in R.*

Next, let the group be G_4. Apply (13) (24) to (9_2). Then

$$x_1-1/x_1=W(x_3).$$

Since this result differs only in sign from (16_2), we get

$$l^2(b-2)+lma+m^2=+1. \tag{20}$$

The unique set of solutions of (13), (14), (20) is now

$$l^2=\frac{a^2}{4A},\quad lm=\frac{a(2+b-a^2)}{4A},\quad m^2=\frac{(2+b-a^2)^2}{4A}.$$

Hence $\sqrt{A}$ must be rational. Formula (18), with the suitable value of $W(x_1)$ inserted, expresses x_3 as a rational function of x_1.

The Galois group, for a domain R, of an irreducible equation (1) *is the group G_4 if, and only if, $\sqrt{A}$ is rational in R.*

4. The last result may be readily verified in various ways. For

$$y_2=x_1x_3+x_2x_4,\quad y_3=x_1x_4+x_2x_3, \tag{21}$$

we have, by (3) and the equation (1),

$$y_2+y_3=b-2,\quad y_2y_3=x_1^2+x_2^2+x_3^2+x_4^2=a^2-2b.$$

Hence* y_2 and y_3 are the roots of

$$y^2-y(b-2)+a^2-2b=0, \tag{22}$$

$$y_2=\tfrac{1}{2}b-1+\sqrt{A},\ y_3=\tfrac{1}{2}b-1-\sqrt{A}. \tag{23}$$

Also, if we set $y_1=x_1x_2+x_3x_4=2$,

$$(y_2-y_3)^2=4A,\ (y_2-2)(y_3-2)=B,\ \Pi(x_i-x_j)^2=\Pi(y_i-y_j)^2=4AB^2.$$

In particular the three y's are distinct numerically. If $\sqrt{A}$ is irrational, the group cannot be G_4, which leaves y_2 and y_3 unaltered. If $\sqrt{A}$ is rational, the group must be contained in G_4, and being transitive is G_4.

Again, by considering the alternating function $\Pi(x_i-x_j)$, we may conclude that the group is contained in G_4 if, and only if, $\sqrt{\Delta}$ and hence also $\sqrt{A}$ is rational.

5. Consider the question of the reducibility of equation (1) in a domain R. A rational root r, other than ±1, would imply a second root $1/r$ and hence a quadratic factor. This is also the case for $r=\pm1$ as the following argument shows. Suppose that E could have a linear and an irreducible cubic factor:

$$E\equiv(x-r)(x^3+\alpha x^3+\beta x-1/r),\quad r,\ \alpha,\ \beta \text{ rational}.$$

Change x to $1/x$ and multiply the new identity by x^4; then

$$E=(1-rx)(1+\alpha x+\beta x^2-\frac{1}{r}x^3),$$

$$E=(x-\frac{1}{r})(x^3-r\beta x^2-r\alpha x-r).$$

Comparing this with the first expression for E, we get

$$r^2=1,\ \beta=-r\alpha.$$

The cubic factor would then vanish for $x=r$ and be reducible.

Hence† if E is reducible in R, we may set

$$E\equiv(x^2+px+r)(x^2+qx+1/r), \tag{24}$$

$$p+q=-a,\ p/r+rq=-a,\ r+1/r+pq=b. \tag{25}$$

If $r=1$, p and q are rational if and only if B, given by (2), is the square of a rational number, $(p-q)^2$.

*Or directly from the resolvent cubic of (1), one root being $y_1=2$.

†From this point on we might use Ferrari's three factorizations of a quartic.

If $r=-1$, then $a=0$ and $\sqrt{(-b-2)}$ must be rational.
Finally, let $r^2\neq 1$. The first two equations (25) give

$$p=\frac{-ar}{r+1},\quad q=\frac{-a}{r+1}.$$

For these values, the third condition (25) becomes

$$y+\frac{a^2}{y+2}=b,\quad y\equiv r+\frac{1}{r}. \tag{26}$$

The former is identical with (22) and has the roots (23). For each y, there are two r's whose rationality depends upon the values of

$$y_2^2-4=C_+,\quad y_3^2-4=C_-, \tag{27}$$

$$C_\pm=\tfrac{1}{2}b^2-2-a^2\pm(b-2)\sqrt{A},\quad C_+.C_-=a^2B. \tag{28}$$

If $a\neq 0$, no one of the four values of r is ± 1 (values not falling under the case being treated). If $a=0$, then $C_-=0$ and we have only to consider the two r's connected with $C_+=b^2-4$.

Theorem. *The necessary and sufficient conditions that* (1) *be irreducible in a domain R are that $\sqrt{B}$ be irrational; that either $\sqrt{A}$ be irrational or, in the contrary case, $\sqrt{C_+}$ and $\sqrt{C_-}$ both irrational if $a\neq 0$, $\sqrt{C_+}$ and $\sqrt{(-b-2)}$ both irrational if $a=0$.*

For the case $a=0$, the conditions reduce to the irrationality of

$$\sqrt{(-b+2)},\ \sqrt{(-b-2)},\ \sqrt{(b^2-4)}.$$

6. Having determined the Galois group of an irreducible equation (1), we pass to the reducible case. First, let B be the square of a number t of the domain R. We may set

$$x_1+x_2-x_3-x_4=t. \tag{29}$$

Since $t^2=B\neq 0$ (§ 1), the group is a subgroup of

$$H_4 : I,\ (12),\ (34),\ (12)(34), \tag{30}$$

the remaining substitutions in (4) replacing t by $-t$. Now

$$x_1+x_2=\tfrac{1}{2}(a+t),\quad x_3+x_4=\tfrac{1}{2}(a-t),$$

so that the rationality of the x's depends upon

(31) $$D_{\pm}=(a\pm\sqrt{B})^2-16,\ D_+.D_-=64A.$$

If $\sqrt{D_+}$ is rational and $\sqrt{D_-}$ is irrational, then x_1 and x_2 alone are rational, and the group is I, (34). If $\sqrt{D_\pm}$ are both irrational and $\sqrt{A}$ rational the group is the common subgroup of H_4, G_4 (see § 8).

7. Finally, let $\sqrt{B}$ be irrational, $\sqrt{A}$ rational, and just one of the pair $\sqrt{C_\pm}$ be rational ($a\neq 0$), one of the pair $\sqrt{C_+}$, $\sqrt{(-b-2)}$ rational ($a=0$). Since $\sqrt{A}$ is rational, the group is a subgroup of G_4. Since $\sqrt{B}$ is irrational the group is not contained in [I, (12)(34)], by (29). By (27),

$$C_+=(x_1x_3-x_2x_4)^2,\ C_-=(x_1x_4-x_2x_3)^2.$$

Hence, for $a\neq 0$, the group is composed of the identity and (14)(23) or (13)(24) according as $\sqrt{C_+}$ or $\sqrt{C_-}$ is irrational.

For $a=0$, $y_2=b$, $y_3=-2$. Set $\alpha=x_1+x_4$. Then

$$-\alpha^2=(x_1+x_4)(x_2+x_3)=2+y_2,\ \alpha=\sqrt{(-b-2)}.$$

If α is irrational, the group is I, (13)(24). If α is rational, then $\alpha\sqrt{B}=2\sqrt{(b^2-4)}$ is irrational ($\alpha=0$ would imply $y_2=y_3$, $\Delta=0$), and hence $x_1x_3-x_2x_4$ is irrational, so that the group is I, (14)(23).

8. The following tabular summary gives the necessary and sufficient conditions on the coefficients a and b of the reciprocal quartic (1) that its Galois group for a domain R will be a specified one of the ten subgroups of G_8. For the abbreviations A, B, C, D, see (2), (28), (31); for the first four groups, see (4), (5), (6), (30).

G_8 : $\sqrt{A}$, $\sqrt{B}$, $\sqrt{(AB)}$ all irrational.
C_4 : $\sqrt{B}$ irrational, $\sqrt{(AB)}$ rational.
G_4 : $\sqrt{B}$ irrational, $\sqrt{A}$ rational, $\sqrt{C_\pm}$ both irrational if $a\neq 0$; $\sqrt{C_+}\equiv\sqrt{(b^2-4)}$ and $\sqrt{(-b-2)}$ both irrational if $a=0$.
H_4 : $\sqrt{B}$ rational, $\sqrt{D_\pm}$ and $\sqrt{A}$ all irrational.
$\{I,\ (12)(34)\}$: $\sqrt{B}$ and $\sqrt{A}$ rational, $\sqrt{D_\pm}$ both irrational.
$\{I,\ (12)\}$: $\sqrt{B}$ and $\sqrt{D_-}$ rational, $\sqrt{D_+}$ irrational.
$\{I,\ (34)\}$: $\sqrt{B}$ and $\sqrt{D_+}$ rational, $\sqrt{D_-}$ irrational.
$\{I\}$: $\sqrt{B}$ and $\sqrt{D_\pm}$ all rational.
$\{I,\ (13)(24)\}$: $\sqrt{B}$ and $\sqrt{C_-}$ irrational, $\sqrt{A}$ rational ($a\neq 0$); $\sqrt{B}$ and $\sqrt{(-b-2)}$ irrational ($a=0$).
$\{I,\ (14)(23)\}$: $\sqrt{B}$ and $\sqrt{C_+}$ irrational, $\sqrt{A}$ rational ($a\neq 0$); $\sqrt{B}$ irrational, $\sqrt{(-b-2)}$ rational ($a=0$).

POSTSCRIPT. After my paper had been printed, I noticed that the discussion on pp. 73-74 could be materially simplified by introducing instead of V the function $U(x_1)=W(x_1).(x_1-1/x_1)$. Then $U=Lv+M$. The

change amounts to throwing the first factor in (12) into the denominator. In place of (13) and (14) we get

$$L(La+2M)=0,\ L^2(2-b)+M^2=4A.$$

According as the group is C_4 or G_4, we have

$$L(a-v)+M=\mp(Lv+M).$$

Hence for C_4, $La+2M=0$, $L^2B=16A$, so that L and M are rational if and only if $\sqrt{(AB)}$ is rational. For G_4, we have $L=0$, $M^2=4A$, so that $\sqrt{A}$ must be rational. In either case the expression for $W(x_1)$ has x_1-1/x_1 in the denominator, but is much shorter than that in the paper.

ON CERTAIN TRANSCENDENTAL FUNCTIONS DEFINED BY A SYMBOLIC EQUATION.*

By PROFESSOR R. D. CARMICHAEL, Anniston, Alabama.

1. Introduction and Definition.

The object of this paper is to define a certain large class of transcendental functions of real variables to which the previous consideration of a geometric subject† gave rise and to point out a few of their general properties. Attention will be confined chiefly to questions of continuity and development in series.

We shall take

(1) $$u=f(x) \text{ and } v=f(y),$$

and put

(2) $$u^v=v^u;$$

thus we have an equation which is satisfied identically when x is put for y; and therefore one solution of the symbolic equation (2) is $y=x$, whatever function is denoted by f. We discard at once this solution as not pertinent to the present discussion.

In order to obtain a second solution, put

(3) $$v=\mu u,$$

*Read before the American Mathematical Society, September 5, 1907.

†Carmichael, The American Mathematical Monthly, Vol. XIII, pp. 221-226, 1906.

where μ is another variable whose law of variation is determined conjointly by equations (2) and (3). A substitution of the latter value of v in the former equation will yield, by an easy reduction,

$$u=f(x)=\mu^{1/(\mu-1)}, \tag{4}$$
$$v=f(y)=\mu^{\mu/(\mu-1)}. \tag{5}$$

Now suppose that the functional equations (4) and (5) are solved, giving

$$x=g(\mu^{1/(\mu-1)}), \tag{6}$$
$$y=g(\mu^{\mu/(\mu-1)}), \tag{7}$$

where g is the function inverse to f. Equations (6) and (7) define y as a function of x; this is the general function to be studied.*

It should be noticed that the preceding discussion does not show that the two given solutions of (2) are the only ones possible. On the other hand it seems indeed quite probable that such a transcendental equation may have other sets of real solutions; and these may give rise to other quite as interesting functions as those of this paper.

2. Continuities and Discontinuities of the Expressions.

In the present section we shall find the conditions under which $\mu^{1/(\mu-1)}$ and $\mu^{\mu/(\mu-1)}=\mu.\mu^{1/(\mu-1)}$ are continuous. It is evident that they are both continuous only when μ itself is continuous.

We shall find that $\mu^{1/(\mu-1)}$ has a continuous graph in the first quadrant. It will now be shown that its graph is discontinuous in the second, third and fourth quadrants; and the nature of the discontinuity will be exhibited.

In the second and third quadrants μ is negative. Consider the value $\mu=-p/q$, where p/q is a fraction in its lowest terms and p and q are both positive. At most one of them can be even. Then

$$u=\mu^{1/(\mu-1)}=\left(-\frac{p}{q}\right)^{\frac{-q}{p+q}}=\left(-\frac{q}{p}\right)^{\frac{q}{p+q}}. \tag{8}$$

Now u is imaginary if both p and q are odd; for then it is equal to an even root of a negative number. And u is real if one of the numbers p and q is odd and the other even. Now consider the five values of μ:

$$-\frac{(2n+1)\,p-2}{(2n+1)q},\quad -\frac{(2n+1)\,p-1}{(2n+1)q},\quad -\frac{p}{q},\quad -\frac{(2n+1)\,p+1}{(2n+1)q},\quad -\frac{(2n+1)\,p+2}{(2n+1)q},$$

*The curve $u^v=v^u$ was studied by Dan Bernoulli and by Euler. The latter gave a parametric representation similar to that here employed, and plotted the curve in the first quadrant. (See Euler, *Introductio*, Vol. 2, p. 294, edition of 1748.) A study from the viewpoint of function theory is thought to be of sufficient importance to justify this presentation. The function well illustrates several important conceptions of the general theory.

where n is a positive integer and one of the numbers p, q is odd and the other even. If p is even, then in the second and fourth fractions, both numerator and denominator are odd, and u is imaginary for each of these values; but to the first and fifth values there evidently correspond real points of the locus. By taking n large at pleasure the first, second, fourth, and fifth values of μ can be brought indefinitely near its third value; and hence on each side of the point corresponding to $\mu=-p/q$ is a point infinitely near to it in the domain of μ, such that to this point corresponds no point on the locus. Moreover this point is seen to lie between two others infinitely near each other such that to each of these corresponds a point of the locus.

If q is even we may consider the set of values

$$-\frac{(2n+1)p}{(2n+1)q+2},\quad -\frac{(2n+1)p}{(2n+1)q+1},\quad -\frac{p}{q},\quad -\frac{(2n+1)p}{(2n+1)q-1},\quad -\frac{(2n+1)p}{(2n+1)q-2}$$

in which p is necessarily odd. A discussion will yield the same conclusion as that obtained when p is even.

These discussions lead readily to the following theorems:

When μ is negative there is in every interval, however small, an infinite number of points on the graph while at the same time there are in the same interval in infinite number of discontinuities. Further, it is clear that *in any such interval $\mu^{1/(\mu-1)}$ is dense, while the points of discontinuity also constitute a dense aggregate.* Finally, *for each point of discontinuity arising as above the function $\mu^{1/(\mu-1)}$ is imaginary.*

In the fourth quadrant u is negative while μ is positive. A discussion similar to that above will lead to the conclusion that *in the fourth quadrant the graph of $\mu^{1/(\mu-1)}$ is infinitely discontinuous.* The discontinuity here arises not by u taking on imaginary values, but by its taking (when real) only positive values for values of μ infinitely near to any value of μ which corresponds to a real point in the fourth quadrant. This is readily seen from a discussion of the following five values of μ:

$$\frac{(2n+1)p-2}{(2n+1)q},\quad \frac{(2n+1)p-1}{(2n+1)q},\quad \frac{p}{q},\quad \frac{(2n+1)p+1}{(2n+1)q},\quad \frac{(2n+1)p+2}{(2n+1)q}.$$

The reader can now readily supply the discussion requisite to establish the fact of infinite discontinuity in the present instance.

The different ways in which the discontinuity arises in these two cases is interesting. A more detailed discussion would show that in the third quadrant the discontinuity arises from the introduction both of imaginary values and of such as these just referred to in the discussion of the locus in the fourth quadrant. It is thence easy to show that along the same branch of the curve in the third quadrant there are now three dense aggre-

gates: (1) the aggregate of real points of the locus; (2) the aggregate of points of discontinuity corresponding to those values of μ for which $\mu^{1/(\mu-1)}$ becomes imaginary; (3) the aggregate of points of discontinuity corresponding to those values of μ for each of which $\mu^{1/(\mu-1)}$ takes on a real positive value but no negative value.

From the preceding discussion it follows that if the graph of $\mu^{1/(\mu-1)}$ possesses a continuous branch it lies in the first quadrant. We proceed to establish the fact of its continuity here.

Let a be any value of μ except $\mu=1$, and let the vicinity of a be the interval $a-h$ to $a+h$. Represent $\mu^{1/(\mu-1)}$ by $F(\mu)$. Then the well known necessary and sufficient condition of continuity that for each $e>0$ there must exist an h such that

$$| F(a+h)-F(a-h) | <e$$

is evidently fulfilled. Hence $F(\mu)$ is continuous everywhere except possibly at $\mu=1$. But as $\mu \doteq 1$ from either side,

$$\mu^{1/(\mu-1)}=[1+(\mu-1)]^{1/(\mu-1)} \doteq e=2.712..;$$

and therefore it follows that the function is continuous at $\mu=1$. Hence it is everywhere continuous. That is, its locus has a continuous branch. As we have seen, this must be in the first quadrant.

Since $\mu^{\mu/(\mu-1)}=\mu.\mu^{1/(\mu-1)}$, the question of the continuity of this function does not require separate consideration; for, since μ is to be taken continuous, it is evident that $\mu^{\mu/(\mu-1)}$ is continuous for all values of μ for which $\mu^{1/(\mu-1)}$ is continuous.

Since $u=\mu^{1/(\mu-1)}$ and $v=\mu^{\mu/(\mu-1)}$, the preceding discussion leads readily to the conclusion that v is a continuous function of u at every point for which both u and v are positive, and equation (2) is satisfied. Likewise u is a continuous function of v under the same limitations. We may then have the theorem:

u and v each is a continuous function of the other throughout the domain of positive rational and irrational numbers.

Now, if in equations (6) and (7) the function denoted by g is continuous, the preceding discussion yields readily the following:

y is a continuous function of x throughout the domain represented by $\mu^{1/(\mu-1)}$ when the domain of μ is all positive rational or irrational numbers.

3. Development in Series.

Resuming equation (4),

$$f(x)=u=\mu^{1/(\mu-1)}, \tag{9}$$

we shall first express μ as an infinite series in terms of u. This will be carried out by the aid of Lagrange's formula and through the help of certain simple substitutions, as follows:*

From (9) we have $u^{\mu-1}=\mu$. Let $u=e^z$. Then $e^{z(\mu-1)}=\mu$. Let $z(\mu-1)=t$; then $e^t=\mu=(\mu-1)+1=t/z+1$. Hence,

$$t=-z+ze^t. \tag{10}$$

If $t=a+b\,\phi(t)$, we have by Lagrange's formula,

$$t=a+b\,\phi(a)+\frac{b^2}{2\,!}\frac{d}{da}[\phi(a)]^2+\ldots+\frac{b^n}{n\,!}\left(\frac{d}{d\,a}\right)^{n-1}[\phi(a)]^n+\ldots \tag{11}$$

Here we have by comparison with (10), $\phi(t)=e^t$, $a=-z$, $b=z$; by a substitution of these values equation (11) reduces to

$$t=-z+ze^{-z}+z^2e^{-2z}+\frac{3z^3}{2!}e^{-3z}+\ldots+\frac{n^{n-2}z^n}{(n-1)\,!}e^{-nz}+\ldots$$

But $\mu=t/z+1$, as we have seen; and therefore equation (9) readily yields

$$\mu=e^{-z}+ze^{-2z}+\frac{3z^2}{2!}e^{-3z}+\ldots+\frac{n^{n-2}z^{n-1}}{(n-1)\,!}e^{-nz}+\ldots$$

Since $z=\log_e u$ and $e^{-z}=1/u$, this becomes

$$\mu=\frac{1}{u}\left[1+\frac{\log_e u}{u}+\frac{3}{2\,!}\left(\frac{\log_e u}{u}\right)^2+\ldots+\frac{n^{n-2}}{(n-1)\,!}\left(\frac{\log_e u}{u}\right)^{n-1}+\ldots\right],$$

and since $\mu\,u=v$, we have

$$v=1+\frac{\log_e u}{u}+\frac{3}{2\,!}\left(\frac{\log_e u}{u}\right)^2+\ldots+\frac{n^{n-2}}{(n-1)\,!}\left(\frac{\log_e u}{u}\right)^{n-1}+\ldots, \tag{12}$$

a development of v in terms of u. Now since u and v enter equation (2) in just the same way, it is clear that the development of u in terms of v may be obtained simply by an interchange of u and v in (12); in other words, the series in (12) has the interesting property that it reverts into the same series in the other variable. Hence, if we denote this function by T, we have

*Cf. THE AMERICAN MATHEMATICAL MONTHLY, Vol. XIII, pp. 18, 72; 1906.

$$u=T(v),\quad v=T(u).$$

Therefore, $v=T[T(v)]=T^2(v).$

But $v=T^{-1}[T(v)],$

whatever function is denoted by T. Hence the above function T is equal to its inverse function. Therefore we may think of T as a functional operation such that

$$T^2=1, \text{ but } T\neq+1 \text{ or } -1.$$

We consider now the question of the convergency of the series in (12). We shall apply the test of the $(n+1)$th term divided by the nth term. To find the limiting value of this quotient, we have the following equations:

$$\frac{(n+1)^{n-1}}{n!}\left(\frac{\log_e u}{u}\right)^n \div \frac{n^{n-2}}{(n-1)!}\left(\frac{\log_e u}{u}\right)^{n-1}$$

$$=\left(\frac{n+1}{n}\right)^{n-1}\frac{\log_e u}{u}=\frac{n}{n+1}\left(1+\frac{1}{n}\right)^n\frac{\log_e u}{u}.$$

But $\underset{n=\infty}{\text{limit}}\ \frac{n}{n+1}\left(1+\frac{1}{n}\right)^n=e.$ Hence the above ratio is less than 1 and the series is convergent for every case for which $\frac{\log_e u}{u}<\frac{1}{e}$. Now it is easy to show that $\frac{\log_e u}{u}$ takes on its maximum value when $u=e$ and that this value is $1/e$. Hence the series in (12) is certainly convergent for every value of u except $u=e$; and for this latter value the preceding discussion gives no answer. For this case, however, it may be shown by other means that $v=e$; for $u=\mu^{1/(\mu-1)}$ takes on the value e only when $\mu=1$, μ being confined to real values; and since $v=\mu u$, $v=u$ for this particular value.

We append here also an interesting expansion of v in exponential form in terms of u. From $u=\mu^{1/(\mu-1)}$ we have $u^{\mu-1}=\mu$; $u^\mu=\mu u$. Whence

$$\mu=\frac{u^\mu}{u}=\frac{u^{\frac{u^\mu}{u}}}{u}.$$

Hence $\mu u=v=u^{\frac{u}{u}^{\frac{u}{u}^{\cdots}}}$.

Evidently the corresponding expansion of u in terms of v is gotten by an interchange of these two variables in the last equation.

PI IN ASIA.

By PROFESSOR GEORGE BRUCE HALSTED, Greeley, Colorado.

In Tokyo at the Buddhist temple of Sengakuji lie buried the forty-seven Ronin, the national heroes of feudal Japan. Just within the gate, in a two-storied building, swords, armor and other relics of these heroes are shown on payment of a fee. By the side of the path leading to the tombs is a well with the inscription, "Here they washed it." No one in Japan needs to be told that "*it*" was the bloody head they were bringing to the grave of their lord, that dead master for whom they considered it the highest privilege thus to forfeit all their lives. The popular reverence for these heroes is still attested not only by the incense perpetually kept burning before their tombs but in stranger fashion by the fresh visiting cards constantly left upon their graves.

All the world knows their exploit, but who knows that one of them, Shigekiyo Matsumura, was the greatest Asiatic mathematician of his age, who in his work *Sanso*, published in 1663, calculated the length of one side of a regular inscribed polygon of 32768 or 2^{15} sides, obtaining

$$0.000095873798655313483$$

and thence for the value of pi 3.141592648, which is true to seven places of decimals, to eight significant figures, while seven is the greatest number that can be obtained by any measurement whatsoever of the most modern science. He also, among many other things, evaluates the volume of the sphere, and treats fully the problem of magic squares, describing the solution of the case where the first 19^2 numbers are arranged in the form of a square. He also treats an analogous problem, not rediscovered in Europe, which may be called the problem of magic circles. solving the case where the number of diameters and the number of circles are eight and sixteen, respectively.

In Europe in 1585 Adriaan Anthoniszoon, father of Adrian Metius, gave for pi 355/113=3.1415929. But this, the most extraordinary of all fractional values, had come a whole millennium earlier in Asia. Yoshio Mikami, under date of January 13, 1908, writes me:

"I have just finished by translation of your *Rational Geometry* (309 pages). To your historical note on pi I have added the following words:

At the end of the Three Kingdoms Lin Huy of Wei gave for pi 157/50 = 3.14. Two centuries afterwards Tsu Chung-tse (425-499) gave the value 355/113. In Japan Seki Kowa employed the same value of pi as Tsu, and this value was given in a printed book of his day."

ON THE RECIPROCAL QUARTIC EQUATION.

By DR. R. L. BERGER.

In this paper it is proposed to determine the Galois group of the reciprocal quartic equation,

$$x^4-ax^3+bx^2-ax+1=0\ldots(1),$$

for the domain of rationality $R(1)$. We shall establish the conditions for which the group of the equation is transitive, and hence the condition that the equation is irreducible; and consider the possible intransitive groups when these conditions are not fulfilled.

Calling the roots of (1) α_0, α_1, β_0, β_1; α_0 the reciprocal of β_0, and α_1 the reciprocal of β_1,

$$\alpha_0\beta_0=\alpha_1\beta_1=1\ldots(2).$$

If we assume that the roots of (1) are distinct, the Galois group of (1) is a subgroup of the group,

$$G_8\equiv[1;\ (\alpha_0\beta_0);\ (\alpha_1\beta_1);\ (\alpha_0\beta_0)(\alpha_1\beta_1);\ (\alpha_0\alpha_1)(\beta_0\beta_1);\ (\alpha_0\beta_1)(\alpha_1\beta_0);\ (\alpha_0\alpha_1\beta_0\beta_1);\ (\alpha_1\alpha_0\beta_1\beta_0)].$$

For, any substitution not in G_8 leaves the relation (2) unaltered only if the equation (1) has a pair of equal roots. The group G_8 has only two transitive subgroups, viz:

$$C_4\equiv[1;\ (\alpha_0\alpha_1\beta_0\beta_1);\ (\alpha_1\alpha_0\beta_1\beta_0);\ (\alpha_0\beta_0)(\alpha_1\beta_1)],$$
$$\text{and } G_4\equiv[1;\ (\alpha_0\alpha_1)(\beta_0\beta_1);\ (\alpha_0\beta_1)(\alpha_1\beta_0);\ (\alpha_0\beta_0)(\alpha_1\beta_1)].$$

Hence, if we impose such conditions upon the coefficients of (1) that its group is either G_8, G_4, or C_4 we have the necessary and sufficient conditions that the equation be irreducible. For this purpose it is necessary to compute the values in terms of the coefficients of functions belonging to each of the following subgroups of G_8:

$$G_4;\ C_4;\ H_4\equiv[1;\ (\alpha_0\beta_0);\ (\alpha_1\beta_1);\ (\alpha_0\beta_0)(\alpha_1\beta_1)].$$
$$G'_2\equiv[1;\ (\alpha_0\alpha_1)(\beta_0\beta_1)];\ G''_2\equiv[1;\ (\alpha_0\beta_1)(\alpha_1\beta_0)].$$

From the equation (1); $\alpha_0+\beta_0+\alpha_1+\beta_1=a\ldots(3)$,
$$(\alpha_0+\beta_0)(\alpha_1+\beta_1)=b-2\ldots(4).$$

By means of (3) and (4) we easily find;

$\phi \equiv (\alpha_0+\beta_0)-(\alpha_1+\beta_1)=\sqrt{[2-b+\tfrac{1}{4}a^2]}$, belonging to H_4... (5),

$\psi \equiv (\alpha_0-\beta_0)(\alpha_1-\beta_1)=\sqrt{[(1+\tfrac{1}{2}b)^2-a^2]}$, belonging to G_4... (6),

$\phi.\psi$ belonging to C_4... (7).

With the aid of (5) and (6);

$\phi' \equiv \alpha_0+\alpha_1=\sqrt{\{\tfrac{1}{4}a^2-(1+\tfrac{1}{2}b)-\sqrt{[(1+\tfrac{1}{2}b)^2-a^2]}\}}$, belonging to G'_2... (8),

$\phi'' \equiv \alpha_0+\beta_1=\sqrt{\{\tfrac{1}{4}a^2-(1+\tfrac{1}{2}b)+\sqrt{[(1+\tfrac{1}{2}b)^2-a^2]}\}}$, belonging to G''_2... (9),

$\phi''' \equiv (\alpha_0-\beta_1)^2=\sqrt{(b^2-4)}$ (when $a=0$), belonging to G''_2... (10).

The values for ϕ and ψ are given up to a rational factor; those for ϕ', ϕ'', ϕ''' up to a rational term.

In the study of the cases where the group is intransitive we shall make use of the two additional functions:

$x_0 \equiv \alpha_0-\beta_0=+\sqrt{\{\tfrac{1}{2}a^2-(b+2)+a\sqrt{[\tfrac{1}{4}a^2+2-b]}\}}$,

belonging to $H_2 \equiv [1; (\alpha_1\beta_1)]$... (11),

$x_1 \equiv \alpha_1-\beta_1=+\sqrt{\{\tfrac{1}{2}a^2-(b+2)-a\sqrt{[\tfrac{1}{4}a^2+2-b]}\}}$,

belonging to $H'_2 \equiv [1; (\alpha_0\beta_0)]$... (12).

From the definition of the Galois group of an equation for a domain of rationality R it follows that if a rational function of the roots of the equation belonging to a group H has a value not in R the group of the equation is not contained in H. Hence we need to consider subgroups of a group H only when a function belonging to H has its value in R.

The group of (1) will depend on the character of the functions (5)-(12) and we have in the following three cases the conditions for which the group G is transitive:

1) ϕ irrational, ψ irrational, and $\phi.\psi$ irrational.

2) ϕ irrational, ψ irrational, and $\psi.\phi$ rational.

$\therefore G=C_4$ or a subgroup of C_4. But its subgroups are excluded as they are subgroups of H_4 to which ϕ belongs. $\therefore G=C_4$.

3) ϕ irrational, ψ rational. Then $G=G_4$, G'_2 or G''_2.

We distinguish two cases:

I) $a\neq 0$. If ϕ' and ϕ'' are irrational, *i. e.*, if the two values $+\sqrt{\{\tfrac{1}{4}a^2-(1+\tfrac{1}{2}b)\pm\sqrt{[(1+\tfrac{1}{2}b)^2-a^2]}\}}$ are irrational, G'_2 and G''_2 are excluded, and $G=G_4$.

II) $a=0$. In this case $\phi''=0$ and we use $\phi'''=\sqrt{(b^2-4)}\neq 0$ to exclude G'_2. $\phi'''\neq 0$, since $b=2$ makes ϕ rational and $b=-2$ makes the equation 1) have a pair of equal roots. If then ϕ' is irrational, *i. e.*, in this case, $\sqrt{(-2-b)}$ is irrational and if $\sqrt{(b^2-4)}\equiv\phi'''$ is irrational, $G=G_4$.

These are the only cases in which the group G is transitive. Therefore we conclude that the necessary and sufficient conditions for the irreducibility of (1) are, if $a \neq 0$,

I) $\sqrt{(2-b+\frac{1}{4}a^2)}$ irrational, and $\sqrt{[(1+\frac{1}{2}b)^2-a^2]}$ irrational.

II) $\sqrt{(2-b+\frac{1}{4}a^2)}$ irrational, $\sqrt{[(1+\frac{1}{2}b)^2-a^2}$ rational, and $\sqrt{\{\frac{1}{4}a^2-(1+\frac{1}{2}b)\pm\sqrt{[(1+\frac{1}{2}b)^2-a^2]\}}}$ irrational.

III) When $a=0$, $\sqrt{(-2-b)}$, $\sqrt{(2-b)}$, $\sqrt{(b^2-4)}$ all irrational. When these conditions are not fulfilled the equation is reducible and its group may be any of the intransitive subgroups of G_8. By examining the values of the set of functions (5)-(12) for any particular case the groups G can be determined. The following cases arise:

1) ϕ rational, ψ irrational. Hence, $G=H_4$, H_2, or H'_2, dependent upon rational or irrational nature of x_0, x_1. Since ψ is irrational, x_0 and x_1 are not both rational. Therefore $G \neq G_1$.

2) ϕ rational, ψ rational. Hence, $G=G_2 \equiv [1; (\alpha_0\beta_0)(\alpha_1\beta_1)]$, the greatest common subgroup of G_4, H_4 and C_4, or if all remaining functions are rational, then $G=G_1$.

3) ϕ irrational, ψ rational. Hence, $G=G'_2$ or G''_2, according as ϕ' or ϕ'' is rational. The case ϕ', ϕ'' both irrational has been included in the previous discussion.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

GEOMETRY.

322. Proposed by FRANK LOXLEY GRIFFIN, S. M., Ph. D., Instructor in Mathematics, Williams College.

Find all surfaces such that the normal lengths intercepted by the three coordinate planes are in constant ratios for all points.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

$$\left.\begin{aligned} n-y+\frac{dz}{dy}(s-z)&=0,\\ m-x+\frac{dz}{dx}(s-z)&=0,\end{aligned}\right\}$$ is the equation to the normal, where (x, y, z) is the point on the surface.

$(x+zdz/dx,\ y+zdz/dy,\ 0)$, where the normal meets the xy plane.

$(x-ydy/dx,\ 0,\ ydy/dz+z)$, where the normal meets the xz plane.

$(0,\ y-xdx/dy,\ xdx/dz+z)$, where the normal meets the yz plane.

Then

$$\pm\frac{x\sqrt{[1+(dx/dy)^2+(dx/dz)^2]}}{y\sqrt{[1+(dy/dx)^2+(dy/dz)^2]}}=a=\pm\frac{xdx}{ydy}\ldots(1).$$

$$\pm\frac{x\sqrt{[1+(dx/dy)^2+(dx/dz)^2]}}{z\sqrt{[1+(dz/dx)^2+(dz/dy)^2]}}=b=\pm\frac{xdx}{zdz}\ldots(2).$$

$$\pm\frac{y\sqrt{[1+(dy/dx)^2+(dy/dz)^2]}}{z\sqrt{[1+(dz/dx)^2+(dz/dy)^2]}}=c=\pm\frac{ydy}{zdz}\ldots(3).$$

$\therefore aydy=xdx,\ bzdz=xdx,\ czdz=ydy,\ aydy+bzdz+czdz=2xdx+ydy.$

$\therefore 2x^2=(b+c)z^2+(a-1)y^2+D.$

$\therefore x^2=Ay^2+Bz^2+D$ are the surfaces satisfying the conditions.

The equation could be put in the form $x^2=ay^2-acz^2+D=0$, where $ac=-b$, as is the case from (1), (2), and (3).

323. Proposed by W. J. GREENSTREET, Marling School, Stroud, England.

S, S' are the foci of two co-vertical parabolas A and B, the axes of which are at right angles. Draw the circle K on SS' as diameter. K is cut in D and E by a straight line parallel to the axis of A such that S' lies midway between it and that axis. Show that the lines $S'D$, $S'E$ are parallel to the two tangents to A which are normals to B.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

The problem as stated is not true.

Let $y^2=4ax$, $x^2=4by$, be the parabolas A, B; $y=mx+a/m$=tangent to A; $y-y'=m(x-x')$ the normal to B.

Since $x'^2=4by'$, $m=-2b/x'$ or $x'=-2b/m$, $y'=x'^2/4b=b/m^2$.

$\therefore y=mx+2b+b/m^2$=normal to B.

But $y=mx+a/m$ and $y=mx+2b+b/m^2$ are the same line.

$\therefore a/m=2b+b/m^2$.

$\therefore m=\frac{1}{4b}[a\pm\sqrt{(a^2-8b^2)}]$, which is true for $a\gtrless 2\sqrt{2}\,b$.

$x^2-ax+y^2-by=0$ is equation to K, $y=2b$=line parallel to axis of A.

$\therefore x^2-ax+2b^2=0$ or $x=\frac{1}{2}[a\pm\sqrt{(a^2-8b^2)}]$.

$\therefore y=\dfrac{2bx}{a\pm\sqrt{(a^2-8b^2)}}+b$ are the equations to $S'D$, $S'E$.

$\therefore$ If m were twice as great, $S'D$, $S'E$ would be perpendicular to the two tangents to A which are normals to B.

The lines through S' parallel to the two tangents are given by the equation

$$y=\frac{1}{4b}[a\pm\sqrt{(a^2-8b^2)}]x+b.$$

This line intersects K in the points

$$\left[-\frac{6ab^2 \pm 2b^2\sqrt{(a^2-8b^2)}}{a^2+4b^2 \pm a\sqrt{(a^2-8b^2)}}, \quad -\frac{a^2b-12b^3 \pm ab\sqrt{(a^2-8b^2)}}{a^2+4b^2 \pm a\sqrt{(a^2-8b^2)}}\right],$$

the plus and minus signs to be used together.

$$y=\frac{8abx}{a^2+4b^2}+12b^3-a^2b+\frac{48a^2b^3}{a^2+4b^2+a\sqrt{(a^2-8b^2)}}$$

is the line through these points. The tangents to A parallel to $S'D$, $S'E$ are

$$y=\frac{2bx}{a\pm\sqrt{(a^2-8b^2)}}+\frac{a[a\pm\sqrt{(a^2-8b^2)}]}{2b}.$$

This line meets $x^2=4by$ in

$$x_1=\frac{4b^2 \pm \sqrt{\{16b^4+2a[a\pm\sqrt{(a^2-8b^2)}]^3\}}}{a\pm\sqrt{(a^2-8b^2)}}=r.$$

The tangent at this point makes an angle with the axis of abscissas whose tangent is $x_1/2b$. As this does not equal $-1/m$ the problem, as stated, is not true.

CALCULUS.

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252. Proposed by J. H. MEYER, S. J., Augusta, Ga.

Supposing the arc of a semi-circle to be stretched out into a straight line, and an indefinite number of perpendiculars erected on it, each equal to the versed sine of the corresponding arc; what would be the length of the curve traced out by the tops of the perpendiculars?

Solution by CHAS. O. GUNTHER, Acting Professor of Mathematics, Stevens Institute of Technology, Hoboken, N. J.

Assuming a as the radius of the circle, the equation of the curve is $y=\text{vers } x/a$, and the required length of the curve is given by the expression

$$s=2\int_0^{\pi a/2}\left(1+\frac{\sin^2 x/a}{a^2}\right)dx=2\int_0^{\pi a/2}\sqrt{(a^2+1-\cos^2 x/a)}\,\frac{dx}{a}.$$

Let $\cos x/a=\sin\theta$; then

$$s=2\sqrt{(a^2+1)}\int_0^{\frac{1}{2}\pi}\sqrt{1-\left(\frac{1}{\sqrt{(a^2+1)}}\right)^2\sin^2\theta}\,d\theta=2\sqrt{(a^2+1)}\,E\left(\frac{1}{\sqrt{(a^2+1)}},\ \tfrac{1}{2}\pi\right).$$

Also solved by G. B. M. Zerr.

253. Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.

Find the maximum number of real points of inflection in each of the quartic curves $y^2=\alpha x^4 \pm x^2+\beta$, and find the necessary and sufficient relations between α and β for the existence of this number of points of inflection.

Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

$y=\sqrt{[\alpha x^4 \pm x^2+\beta]}$. Then $\frac{dy}{dx}=\frac{2\alpha x^3 \pm x}{\sqrt{[\alpha x^4 \pm x^2+\beta]}}$.

$$\frac{d^2y}{dx^2}=\frac{2\alpha^2x^6 \pm 3\alpha x^4(2\alpha-1)+6\alpha^2\beta x^2 \pm \beta}{(\alpha x^4 \pm x^2+\beta)^{\frac{3}{2}}}=0.$$

$$\therefore x^6 \pm \frac{3(2\alpha-1)}{2\alpha}x^4+3\beta x^2 \pm \frac{\beta}{2\alpha^2}=0.$$

Let $x^2=z$, $\frac{3(2\alpha-1)}{2\alpha}=a$, $3\beta=b$, and $\frac{\beta}{2\alpha^2}=c$. $\therefore z^3 \pm az^2+bz \pm c=0$.

Let $z=u \mp \frac{1}{3}a$. Then $u^3-(\frac{1}{3}a^2-b)u \mp (\frac{ab}{3}-\frac{2}{27}a^3-c)=0$, or $u^3-Au \mp B=0$.

$\therefore$ When $A^3/B^2<27/4$, there are three real unequal roots.

$$\therefore \frac{[(2\alpha-1)^2-4\alpha^2\beta]^3}{[6\alpha^2\beta(2\alpha-1)-2\alpha\beta-(2\alpha-1)^3]^2}<1.$$

Then $y^2=\alpha x^4+x^2+\beta$ can have six real points of inflection, while $y^2=\alpha x^4-x^2+\beta$ can have but four. The values of α and β can reduce this number in both cases.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

122. Proposed by DR. L. E. DICKSON, The University of Chicago.

If p is prime, $(p^4-1)(p^2-1)$ has no factor of the form $1+p^3x$, $x>0$, if $p>2$; $(p^6-1)(p^4-1)(p^2-1)$ has no factor of the form $1+p^5x$, $x>0$.

Solution by H. S. VANDIVER, Bala, Pa.

I. Assume that $(p^4-1)(p^2-1)$ is divisible by $1+p^3x$ where x is not zero, p *any* integer, not zero. By inspection we see that there exists a positive integer y, not zero, such that

$$(p^4-1)(p^2-1)=(1+p^3x)(1+p^5y).$$

Now write the left hand member in the scale of notation whose radix is p. Then

$$p^5(p-1)+p^4(p-2)+p^3(p-1)+p^2(p-1)+1 \\ =1+p^3x+p^2y+p^5xy.$$

From which it is evident that $xy \geqslant p-1$, and x and y are each less than p. The right hand member is therefore also expressed in the p-scale of notation, and the two expressions are identical. Whence, equating coefficients of p^4, we obtain $p=2$, and $(2^4-1)(2^2-1)=(1+2^3)(1+2^2)$. Otherwise the assumed equality is impossible. Note that p is not necessarily prime.

II. Assume that $(p^6-1)(p^4-1)(p^2-1)$ has a factor of the form $1+p^5x$, $x>0$, then there is necessarily an integer $y>0$, such that

$$P=(p^6-1)(p^4-1)(p^2-1)=(1+p^5x)(-1+p^2y)\ldots(1).$$

Whence, by expansion and division by p^2,

$$y=1+p^2+p^3x+p^{10}-p^8-p^6-p^5x\ldots(2),$$

and y is therefore of the form $1+p^2+p^3M$.

Since $y>0$, then $M \geqq 0$. If $M=0$, $y=p^2+1$, and from (2),

$$x=p^3-\frac{2p^5}{p^4+p^2-1}.$$

Now p^5 is prime to p^4+p^3-1, and this is contrary to the hypothesis that x is an integer. Assume, then, $M>0$. In this case, evidently $y>p^3$. As a consequence, $x<p^2$, for if $x \geqq p^2$ then from (1), $P>(1+p^2)(-1+p^5)$, and by expanding and dividing by p^2, the result may be written $p^8-1+p^5(p-1)+p^2(p-1)<0$, which is absurd. In a similar way, it may be shown from (1), that $y<p^5$, for all values of x, not zero.

From (2), $y-(1+p^2+p^3x)$ is divisible by p^5. Now since $x<p^2$, $1+p^2+p^3x<p^5$, and since $y<p^5$, we must have $y=p^3x+p^2+1$. Comparing with (2), we obtain

$$p^5-p^3-p=x(p^3x+p^2+1),$$

and evidently $x \equiv 0 \pmod{p}$. Hence the right hand member is greater than p^5. But the left hand member is less, so (1) is impossible. Note again that p is not necessarily prime.

PROBLEMS FOR SOLUTION.

ALGEBRA.

298. Proposed by J. F. LAWRENCE, A. B., Professor of Mathematics, Stillwater, Okla.

If $\alpha, \beta, \gamma, \ldots$, are the roots of the equation $\sin mx - nx\cos mx = 0$, prove that $\tan^{-1}\frac{x}{\alpha} + \tan^{-1}\frac{x}{\beta} + \ldots + \tan^{-1}\frac{x}{\nu} = 0$.

GEOMETRY.

331. Proposed by DAVID F. KELLEY, New York, N. Y.

To find the area of a parabolic sector, by a hitherto unpublished method.

CALCULUS.

255. Proposed by A. H. HOLMES, Brunswick, Maine.

Evaluate $\int_0^{\frac{1}{2}a} \frac{dx}{\sqrt{[2ax - x^2}\sqrt{(a^2 - x^2)]}}$.

256. Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.

Solve the differential equation, $(1+y+2xy)\,dx + x(1+x)\,dy = 0$.

MECHANICS.

215. Proposed by HENRY WRITT, Genoa Junction, Wisconsin.

Suppose two centers of attractive forces A and B having a ratio $1 : 330{,}000$, and influence reducing as the second power of the distance, *i. e.*, R^{-2}. Then there is a point, P, on the line joining A and B, where $\frac{1}{AP^2} = \frac{330{,}000}{BP^2}$, or $1 : 575$, nearly. At this point the attractions are equal but opposite in direction along AB. It is proposed to find the surface through the point P which is the locus of the direction of the resultant of the two forces directed towards A and B, *i. e.*, the locus of the diagonals of the minimum parallelogram of forces.

NOTES AND NEWS.

Dr. Gilbert A. Bliss, of Princeton University, will offer courses at the University of Chicago during the next Summer quarter.

Dr. Joseph V. Collins's *Elementary Practical Algebra*, published by the American Book Company, is now going through the press.

Professor E. H. Moore is spending the year in Europe. He presented a paper at the International Congress of Mathematicians held in Rome, Italy, early in April.

Professor H. S. White, of Vassar College, will give the courses, at the University of Chicago in the next Summer quarter, which were to have been given by Professor Maschke.

The Chicago Section of the American Mathematical Society held its Spring meeting at the University of Chicago on April 17 and 18. There were twenty-three papers presented, among which were several to be offered for publication in the MONTHLY.

At the University of Chicago, Assistant Professors Kent Loves and Frank R. Moulton, of the Department of Mathematical Astronomy, and Herbert E. Slaught and J. W. A. Young, of the Department of Mathematics, have been promoted to Associate Professorships.

In the Summer School of the University of Pennsylvania, the following courses in Advanced Mathematics will be offered: Definite Integrals, by Professor Schwatt; Differential Equations, by Professor Safford; Theory of Functions of a Complex Variable, by Professor Hallett.

At the next meeting of the National Educational Association in Cleveland next July, the Mathematical Section will be in charge of Professor H. E. Slaught, of the University of Chicago. The speakers will be Professor T. E. McKinney, of Wesleyan University, Middletown, Conn.; Mr. William Betz, of the East High School, Rochester, N. Y.; Professor John C. Stone, Michigan State Normal College, Ysilanti, Mich.; and Mr. Richard S. Beardsley, of the Englewood High School, Chicago.

The following courses are announced for the Summer quarter at the University of Chicago, each extending through the entire quarter and meeting either four or five hours per week: By Professor H. S. White, Higher Plane Curves, and Elliptic Functions. By Professor H. E. Slaught, Advanced Integral Calculus, and Analytical Geometry. By Professor J. W. A. Young, Differential Calculus, and Pedagogy of Secondary Mathematics. By Professor L. E. Dickson, Theory of Equations, and Plane Trigonometry. By Professor G. A. Bliss, Differential Equations from the Standpoint of Continuous Groups, and Integral Calculus. By Dr. A. C. Lunn, Analytic

Mechanics, and College Algebra. By Professor G. W. Myers, in the College of Education, Pedagogy of Secondary Mathematics, and Pedagogy of Elementary School Mathematics.

BOOKS.

The Theory of Functions of a Real Variable and the Theory of Fourier's Series. By W. E. Hobson, Sc. D., F. R. S., Fellow of Christ's College and Stokes Lecturer in Mathematics in the University of Cambridge. Royal 8vo. Cloth, xvi+772 pages. Price, $6.50. Published in America by G. P. Putnam's Sons, New York City.

This work presents in a connected form, and renders more easily accessible than hitherto, the chief results of the investigations, in the Theory of Functions of a Real Variable, made by Cantor, Dedekind, Weierstrasse, and others. Gaps in the various theories have been filled up, proofs of many of the theorems have been simplified, and some theorems have been given in a more general form than that in which they were originally discovered.

The table of contents will give some idea of the extensive treatment of some of the most interesting developments in modern mathematics. Thus, Chapter I contains a discussion of Number, and includes a full account of the theories of Real Number, due to Cantor and Dedekind; Chapter II contains an exposition of the theory of sets of points, and includes an account of transfinite cardinal and ordinal Arithmetic; Chapter III deals pretty fully with the general theory of aggregates; Chapter IV discusses the main properties of functions in relation to continuity, discontinuity, etc., including investigations of the properties of important classes of functions; Chapter V is devoted to a discussion of the Foundations of the Integral Calculus, as based upon Riemann's definition of a definite integral, and its extension. Also in this chapter is given an account of the development of the Integral Calculus from the view point of Lebesgue's new definition of the definite integral; Chapter VI is concerned with functions defined as the limits of sequences of functions, and contains an account of the principal properties of functions represented by series, and a discussion of important matters connected with the modes of convergence of series through whole intervals, or in the neighborhood of particular points. An account of the very general results recently obtained by Baire, relating to the representability of functions by means of series, will be found in this Chapter. Chapter VII is devoted to the theory of Fourier's Series.

The work constitutes the most thorough, comprehensive, and satisfactory treatment of the Theory of Functions of Real Variables that has thus far appeared in the English language. B. F. F.

High School Algebra.—Advanced Course. By H. E. Slaught, Ph. D., Associate Professor of Mathematics in the University of Chicago, and N. J. Lennes, Ph. D., Instructor in Mathematics in the Massachusetts Institute of Technology. 12mo. Cloth, vii+194 pages. Price, 75 cents. Chicago: Allyn & Bacon.

It is believed that this text book of Algebra, which includes a review of the authors' Elementary Course, contains all the algebra required by most technical and scientific schools. The material of this course is well selected and skillfully arranged. The graph is used, but not to excess. Some books are making too much of the graph, the danger being that students using a text in which over-emphasis is laid on graphics are likely to go off with the authors on the tangent to the graph. B. F. F.

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ON THE GENERALIZATION OF A THEOREM IN SOLID GEOMETRY.

By ROBERT E. MORITZ, Ph. D., The University of Washington, Seattle.

It is shown in solid geometry that the volume of a truncated right triangular prism is equal to the product of the area of its base by one-third the sum of the lateral edges. In symbols,

$$V_3 = B_3\left(\frac{e_1+e_2+e_3}{3}\right), \qquad [1]$$

where B_3 denotes the base; e_1, e_2, e_3 the lateral edges; and V_3 the volume of the truncated prism.

A truncated right quadrangular prism may be divided into two truncated right triangular prisms by passing a plane through a pair of opposite lateral edges, say through e_1 and e_3, where e_1, e_2, e_3, e_4 denote the lateral edges taken in succession. If furthermore B_4 denotes the quadrangular base, B'_3 and B''_3 the respective bases of the component triangular prisms, and V_4 the volume of the truncated quadrangular prism, the application of formula [1] gives

$$V_4 = B'_3\left(\frac{e_1+e_2+e_3}{3}\right) + B''_3\left(\frac{e_3+e_4+e_1}{3}\right). \qquad [2]$$

If the base of the truncated prism is a parallelogram, B'_3 and B''_3 are each equal to one-half of B_4, [2] therefore becomes

$$V_4 = \frac{B_4}{2}\left(\frac{\overline{e_1+e_2+e_3+e_4}+\overline{e_1+e_3}}{3}\right). \qquad [3]$$

But now the upper base is a parallelogram also, and from the fact that the diagonals of this upper base bisect each other it readily follows that